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Relativistic corrections for time and frequency transfer in optical fibres

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Abstract

We derive relativistic corrections for one-way and two-way time and frequency transfer over optical fibres neglecting no terms that exceed 1 ps in time and 10^{-18} in fractional frequency, and estimate their magnitude in typical fibre links. We also provide estimates of the uncertainties in the evaluation of the relativistic corrections due to imperfect knowledge of parameters like the coordinates of the fibre and stations, Earth rotation, or thermal effects of the fibre index and length. The links between Teddington(UK) and Paris(F) as well as Braunschweig(D) and Paris(F), that are currently under construction, are studied as specific examples.

Keywords: Time and frequency transfer, optical fibres, relativistic corrections

(Some figures may appear in colour only in the online journal)

1. Introduction

Atomic clocks have been improving rapidly over the past years and are now reaching stabilities and accuracies of a few parts in 10^{18} in fractional frequency [1, 2]. Applications of such clocks in fundamental physics, geodesy, navigation etc require their comparison over large distances without degrading their performance. At present, no existing satellite or other long (intercontinental) distance comparison method reaches the required level of uncertainty. However, over short to medium distances (10^3 km) optical fibre links have demonstrated performance below 10^{-18} in frequency transfer [3–5] and such fibre networks are therefore one of the key technologies being developed for the application of the new generation of atomic clocks in many fields.

The increasing requirements on time/frequency accuracy and stability pushes the evaluation of phenomena that affect the signal propagation in fibre links, and the models for their correction, to new levels of uncertainty. One ensemble of such corrections are related to the evaluation of relativistic effects for a fibre which is moving with given velocity due to the Earth's surface motions (rotation, tides) and which is exposed to the Earth's gravitational field. Systematic relativistic theory

of time and frequency transfer has been worked out e.g. for the case of satellite transfer so far [6–9].

In our work we present a systematic relativistic description of signal propagation in optical fibres and we derive relativistic corrections with an uncertainty that is sufficient for the new generation of atomic clocks. We provide expressions for one-way and two-way time and frequency transfer neglecting no terms that can exceed 1 ps in time and 10^{-18} in fractional frequency. We then estimate the magnitude of these terms and their uncertainty from imperfect knowledge of the required parameters (fibre and station coordinates, Earth rotation, fibre index variations,...). Finally, we evaluate the expressions for time and frequency transfer examples in Europe, in particular on the PTB-SYRTE and NPL-SYRTE links that are under construction.

In the main part of the text the resulting formulas for relativistic corrections are summarized and the technical details of the derivation can be found in the appendix.

2. Definitions

In this section we define some concepts and quantities which will be used for the formulation of relativistic corrections for time and frequency transfer.

First we consider a signal propagating in an optical fibre and an observer who is not moving with respect to the fibre at a certain fibre point. We suppose that a coordinate velocity of the signal as measured in a local coordinate system of this observer, or more precisely a magnitude of the velocity component which is tangent to the fibre, is given as

$$v = \frac{c}{n_{\text{eff}}} \tag{1}$$

where n_{eff} is an effective refractive index. The formula (1) can be understood as definition of the effective refractive index. Its value should be determined experimentally for a particular fibre. This value can differ from a value of refractive index for free propagation in the medium due to the interaction of the light signal with the walls of the fibre. The value of effective refractive index also depends on the temperature of the fibre, stress in the fibre, the frequency of the signal, and the polarisation of the signal, possibly on fibre bending, etc. For a waveguide, $n_{\text{eff}} = \beta/\beta_0$ where β and β_0 are, respectively, the propagation constants of the wave along the waveguide direction and in vacuum. We will consider in the following that we can describe the signal by a light ray. Therefore we do not describe wave effects such as dispersion, polarisation or interferences explicitly. Nevertheless, waveguide effects are accounted for by the fact that the effective index can change with time and along the fibre, and by the fact that the light ray represents one (transverse) guided mode of the fibre with propagation constant β . In the following text we drop the index eff and we denote the effective refractive index just n .

Throughout this work we use the Geocentric Celestial Reference System (GCRS). This coordinate system is centered in the centre of the mass of the Earth and is non-rotating with respect to distant stars. The GCRS coordinates are denoted (x^0, x^i) where $x^0/c = t$ is the Geocentric Coordinate Time (TCG) and $x^i = (x, y, z)$ are the spatial coordinates of the system. The small Latin indices go from 1 to 3.

Components of the metric in the GCRS coordinates are [10]

$$g_{00} = -1 + \frac{2w}{c^2} - \frac{2w^2}{c^4} + O(c^{-5}) \tag{2}$$

$$g_{0i} = -\frac{4}{c^3}w_i + O(c^{-5}) \tag{3}$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2w}{c^2}\right) + O(c^{-4}) \tag{4}$$

where w and w_i are, respectively, a scalar and a vector potential defined in [10, equation (20)] with convention $w \geq 0$.

We can treat the spatial slices of our spacetime given by $t = \text{const.}$ as Euclidean spaces, similar to Newtonian physics. The metric of these Euclidean spaces can be defined by components δ_{ij} in the spatial GCRS coordinates x^i . The tangent vectors of these spaces (analogs of vectors in Newtonian physics) we denote in bold and their scalar product given by the Euclidean metric by dot. We denote \mathbf{e}_i the coordinate basis of the coordinates x^i and thus we have $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

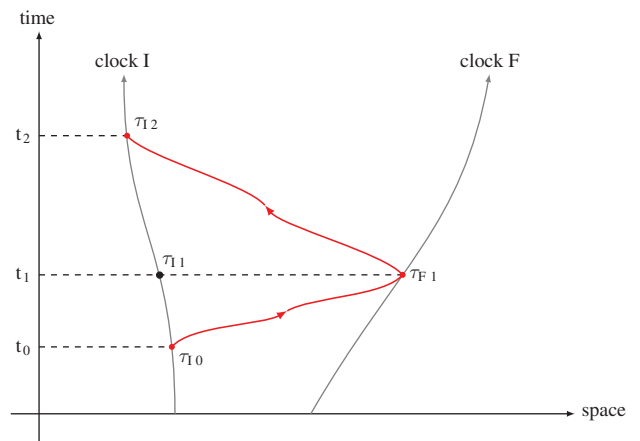


Figure 1. The signal leaves observer I at coordinate time t_0 corresponding to proper time τ_{00} of clock I , is reflected from observer F at coordinate time t_1 corresponding to proper time τ_{F1} of clock F and τ_{11} of clock I , and finally is received by the observer I at coordinate time t_2 corresponding to proper time τ_{12} of clock I .

The fibre is considered to be a one-dimensional object. Its trajectory in spacetime is therefore parametrised by two parameters—the coordinate time t and a parameter λ which has a unique value for a given fibre element and is growing from an initial point of the fibre which we denote I to the final point of the fibre which we denote F . The fibre trajectory is given parametrically as

$$x^i = x^i(t, \lambda). \tag{5}$$

For a fixed value of λ this equation gives a trajectory of a corresponding fibre element. We can define a velocity vector field of the fibre in the GCRS frame and a tangent vector field of the fibre with parameter λ as

$$\mathbf{v} = \frac{\partial x^i}{\partial t} \mathbf{e}_i, \quad \mathbf{s}_\lambda = \frac{\partial x^i}{\partial \lambda} \mathbf{e}_i. \tag{6}$$

3. Time transfer

We consider a signal emitted from observer I at coordinate time t_0 which corresponds to proper time τ_{00} of clock I . The signal is then received by observer F at coordinate time t_1 which corresponds to proper time τ_{F1} of clock F (see figure 1). The ‘pseudo-time-of-flight’ $\tau_{F1} - \tau_{00}$ is obtained from measurements. Then a signal is sent from observer F at coordinate time t_1 and received by observer I at coordinate time t_2 which corresponds to proper time τ_{12} of clock I . This signal can be either the same signal reflected or another signal which is synchronously sent when the one-way signal is received by observer F . We call this set-up the Λ -configuration. The ‘pseudo-time-of-flight’ $\tau_{12} - \tau_{F1}$ is also obtained from measurements. Using the coordinate time synchronisation convention we define $\tau_{11} = \tau_I(t_1)$.

We denote $\Delta t_+ = t_1 - t_0$ the coordinate time of signal propagation from I to F and $\Delta t_- = t_2 - t_1$ the coordinate time of signal propagation from F to I .

It will be convenient to introduce a special choice of the fibre parameter λ , which is given by a rest length of the fibre

from its initial point I at time t_1 . We denote this parameter l (see appendix A.2 and formula (A.11) for a precise definition). The range of this parameter is $l \in [0, L]$ with L being the total rest length of the fibre at the time t_1 .

For the coordinate propagation times we obtain the following formula up to c^{-3} order (see appendix A for derivation) which gives all the terms larger than 1 ps

$$\Delta t_{\pm} = \frac{1}{c} \int_0^L n \, dl \pm \frac{1}{c^2} \int_0^L \mathbf{v} \cdot \mathbf{s}_l \, dl + \frac{1}{c^3} \int_0^L n(w + v^2/2) \, dl \quad (7)$$

where $v^2 = \mathbf{v} \cdot \mathbf{v}$ and all quantities in this formula are evaluated at $t = t_1^3$.

For a particular case of a 1000 km long optical fibre with refractive index of 1.5 we obtain the following numerical estimates of the terms in (7). The leading Newtonian term (the c^{-1} term) has a value of 5 ms. If the fibre is located along the equator the value of c^{-2} term is ± 5 ns. This term corresponds to the Sagnac correction, as we will see. The c^{-3} term gives 3 ps at surface of the Earth. This term is a part of the Shapiro correction. Another part appears if we use the Euclidean length instead of the rest length as the fibre parameter (see appendix A for further details). Higher order terms in $1/c$ are negligible for the 1 ps accuracy level. For example the c^{-4} term is of order of 10^{-23} s per one meter of the fibre length.

Analogous corrections also appear for the satellite time transfer (see e.g. [7]). However, in the case of fibres where the signal propagation is not geodesic, the state of the fibre given by its position, velocity and refractive index enters the formulas. This is the difference compared to the satellite transfer, where only the state of the emitter and receiver appears.

The full formula (7) would be used in case of one-way time transfer. However, in practice it is difficult to evaluate the leading term of this formula with sufficient accuracy. Therefore, the two-way time transfer is used which compensates this term.

Using the coordinate synchronisation convention, the desynchronisation of the clocks is defined as difference $\tau_{F1} - \tau_{I1}$. In the case of the two-way time transfer it can be expressed as

$$\tau_{F1} - \tau_{I1} = \frac{1}{2}(\tau_{F1} - \tau_{I0} + \tau_{F1} - \tau_{I2}) + \frac{1}{2}(\Delta\tau_- - \Delta\tau_+) \quad (8)$$

where $\Delta\tau_+ = \tau_{I1} - \tau_{I0}$ and $\Delta\tau_- = \tau_{I2} - \tau_{I1}$. The ‘pseudo-time-of-flights’ $\tau_{F1} - \tau_{I0}$ and $\tau_{I2} - \tau_{F1}$ in (8) are measured and the difference $\Delta\tau_- - \Delta\tau_+$ needs to be computed. The computed term can be approximated as

$$\Delta\tau_- - \Delta\tau_+ \approx \Delta t_- - \Delta t_+ \quad (9)$$

The error of this approximation is several orders below the required 1 ps level for a fibre on the Earth’s surface. Using (7) we obtain

$$\frac{1}{2}(\Delta t_- - \Delta t_+) = -\frac{1}{c^2} \int_0^L \mathbf{v} \cdot \mathbf{s}_l \, dl \quad (10)$$

³ If needed the integrals can be expressed in terms of Euclidean length parameter l_E instead of l using the transformation formula (A.16).

The term $\mathbf{s}_l \, dl = dx^i \mathbf{e}_i$ does not depend on the choice of the fibre parameter (l in this case), and therefore the integral in (10) can be easily transformed if any other parameter (e.g. the Euclidean length) is more convenient.

The main part of the velocity in formula (10) and in the second term of formula (7) comes from the Earth’s rotation. To interpret this term we introduce a new coordinate frame which rotates rigidly with respect to the Euclidean frame of the GCRS system with angular velocity vector $\boldsymbol{\omega}$. We suppose that this rotating frame follows the rotation of the Earth’s surface such that the residual velocity of the fibre in this rotating frame is minimised. We refer to this frame as to a co-rotating frame. We denote the position vector of a fibre point by \mathbf{x} and the velocity vector of a fibre point in the co-rotating frame by \mathbf{v}_R . We have

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x} + \mathbf{v}_R \quad (11)$$

For the scalar product in (10) we get

$$\begin{aligned} \mathbf{v} \cdot \mathbf{s}_l &= (\boldsymbol{\omega} \times \mathbf{x}) \cdot \mathbf{s}_l + \mathbf{v}_R \cdot \mathbf{s}_l \\ &= \boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{s}_l) + \mathbf{v}_R \cdot \mathbf{s}_l \end{aligned} \quad (12)$$

If we write $\boldsymbol{\omega} = \omega \mathbf{o}$ where $\mathbf{o} \cdot \mathbf{o} = 1$, i.e. \mathbf{o} is a unit vector in direction of $\boldsymbol{\omega}$ we can define a quantity

$$A = \frac{1}{2} \int_0^L \mathbf{o} \cdot (\mathbf{x} \times \mathbf{s}_l) \, dl \quad (13)$$

This quantity is the Sagnac area of the fibre, which for certain simple fibre paths can be understood as the area of a surface which is lying on a plane perpendicular to $\boldsymbol{\omega}$ between a projection of the fibre onto this plane and lines connecting the endpoints of this projection with the rotation axis. This area can be positive or negative depending on whether the parameter of the fibre (l in this case) goes along or against the direction of the Earth’s rotation. Using the Sagnac area A we can write

$$\int_0^L \mathbf{v} \cdot \mathbf{s}_l \, dl = 2\omega A + \int_0^L \mathbf{v}_R \cdot \mathbf{s}_l \, dl \quad (14)$$

In this formula the first term is usually much larger than the second one. We can estimate the second term if the velocity \mathbf{v}_R is caused by the Earth’s tides. The Earth’s tides are oscillatory deformations of the Earth’s body with a period of 12h and an amplitude of vertical Earth surface motion of approx. 30 cm leading to a maximal vertical velocity of the Earth’s surface of approx. 0.05 mm s^{-1} . The effect would be maximal for a fibre path following a meridian near 45° latitude. The second term of (14) then gives 0.3 fs contribution to the correction (10) per 1000 km of the fibre in this maximising example.

The phenomena affecting the time transfer are summarized in table 1.

4. Frequency transfer

For one-way frequency transfer we consider that a signal with proper frequency ν_e is emitted from one observer and the same signal is received by a second observer with proper frequency ν_r . We suppose that a phase of the signal which is observed by

Table 1. Effects influencing the time transfer in optical fibres and sizes of the corresponding contributions to the one-way and two-way time transfer formulas.

Effect	Contribution per 1000 km
Length and refractive index of the fibre (Newtonian term; 1-way only)	5 ms
Velocity of the fibre due to the Earth's rotation	5 ns
Velocity of the fibre due to the Earth's tides	0.3 fs
Gravitational plus centrifugal potential on the Earth's surface (1-way only)	3 ps

Note: All contributions are calculated for 1000 km of the fibre length. The values of the second and the third contribution (Sagnac correction) are examples for specific fibre positions where the effect is maximised (see the text for details).

the emitting observer at its proper time τ_e is observed by the receiving observer at its proper time τ_r . This is illustrated in figure 2.

The time τ_r is uniquely given by the time τ_e so we can define a function $\tau_r(\tau_e)$. The ratio of the proper frequencies is then given as

$$\frac{\nu_r}{\nu_e} = \left(\frac{d\tau_r}{d\tau_e}\right)^{-1} = \left(\frac{dt_e}{d\tau_e}\right)^{-1} \frac{d\tau_r}{dt_r} \left(\frac{dt_r}{dt_e}\right)^{-1}. \quad (15)$$

The derivatives $dt_e/d\tau_e$ and $dt_r/d\tau_r$ do not depend on the fibre itself but just on the state of the emitting and receiving observers. They contain the gravitational red shift and second-order Doppler effect corrections. The same terms also appear for different ways of frequency transfer such as the satellite transfer (see e.g. [7]). These terms are investigated in detail in [6] and it is not the purpose of this paper to study them. For a relative accuracy of the frequency transfer of 10^{-18} , these terms contain the Newtonian gravitational potential of the Earth and of external masses, as well as a correction for the non-geodesic barycentric motion of the Earth. Recently these terms have been determined with an accuracy of 10^{-17} [11]. At present, a combination of terrestrial data sets with up-to-date satellite gravity field models allows the calculation of these terms with accuracies of a few parts in 10^{18} , provided that high-resolution and high-quality terrestrial and satellite data are utilised [12]. Related research is tackled in the frame of the project ITOC (International Timescales with Optical Clocks) [13]. On the other hand, the frequency comparison can be used to measure directly the geopotential difference between two clocks, a technique called chronometric leveling [14]. A large-scale demonstration of chronometric leveling is also one of the task of the project ITOC.

In this paper we focus on how the frequency transfer is affected by processes in the fibre itself. These effects are contained in the dt_r/dt_e term of (15). This term includes e.g. influences of changing refractive index, changing fibre length due to thermal expansion or changing Sagnac area due to fibre motion.

We denote $T(t,l)$ a temperature of the fibre as a function of time and location and α the linear thermal expansion coefficient of the fibre. Again we consider a signal sent from I at time t_0 , reflected in F at time t_1 and received back by I at time

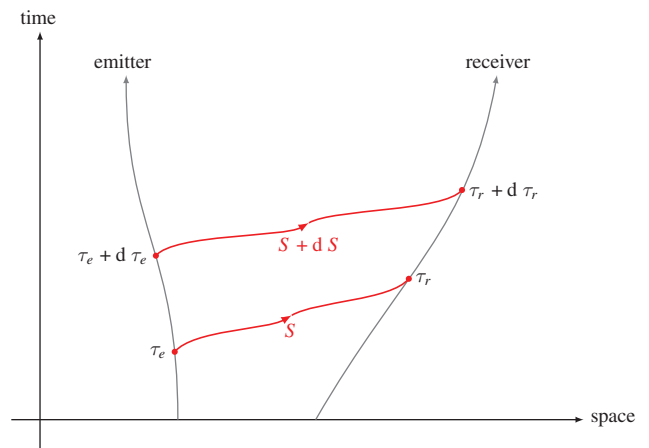


Figure 2. Two clocks are measuring proper time along their trajectory. One signal with phase S is emitted at proper time τ_e , and another one with phase $S + dS$ at time $\tau_e + d\tau_e$. They are received, respectively, at time τ_r and $\tau_r + d\tau_r$. The proper frequency measured by the emitter/receiver is, respectively: $\nu_{e/r} = \frac{1}{2\pi} \frac{dS}{d\tau_{e/r}}$.

t_2 . The dt_r/dt_e term is then given by the following formula up to c^{-2} order (see the appendix A for derivation)

$$\begin{aligned} \frac{dt_{r\pm}}{dt_e} &= 1 + \frac{1}{c} \int_0^L \left(\frac{\partial n}{\partial t} + n\alpha \frac{\partial T}{\partial t} \right) dl \\ &\pm \frac{1}{c^2} \int_0^L \frac{\partial \mathbf{v} \cdot \mathbf{s}_l}{\partial t} dl \end{aligned} \quad (16)$$

where $+$ sign relates to the propagation from I (emitter) to F (receiver) and $-$ sign to the propagation from F (emitter) to I (receiver). All quantities in the formula are evaluated at the time $t = t_1$. The formula includes all terms larger than 10^{-18} . The order of the c^{-3} term is estimated to be 10^{-28} per one metre of the fibre length due to thermal effects. The c^{-1} term of (16) corresponds to the first order Doppler effect and the c^{-2} term is a time derivative of the Sagnac correction. Analogous terms can be found also in the satellite frequency transfer [7], however, some differences occur. For example the Doppler shift for fibres does not depend on velocities of the emitter and receiver but on expansion and change of the refractive index of the fibre.

Now we look to the c^{-1} term of (16) in more detail and we estimate its value. The change of the refractive index with time can be caused by various phenomena. For our estimation we consider a change of n due to temperature variation, i.e. $\partial n/\partial t = (\partial n/\partial T) \cdot (\partial T/\partial t)$. The values of the linear thermal expansion coefficient and temperature derivative of the refractive index, based on [15], can be estimated as

$$\alpha = 8 \times 10^{-7} \text{ K}^{-1}, \quad \frac{\partial n}{\partial T} = 10^{-5} \text{ K}^{-1}. \quad (17)$$

The time derivative of the temperature can be estimated based on the experiment described in [16]. Its maximal value in this experiment was

$$\frac{\partial T}{\partial t} \approx 4 \times 10^{-6} \text{ K s}^{-1}. \quad (18)$$

If we consider a 1000 km long fibre with $n = 1.5$ we obtain a value of 2×10^{-13} for the c^{-1} term of (16).

In the case of one-way frequency transfer the full formula (16) is needed; however, in practice it is difficult to determine the c^{-1} term with sufficient accuracy. Therefore, it needs to be actively compensated or suppressed by means of two-way frequency transfer. The c^{-2} term of (16) appears also in the two-way transfer and it is discussed in more detail below.

During two-way frequency transfer the observer I emits a signal with proper frequency ν_{I0} . The signal is received at observer F and is immediately transponded back to observer I , where a proper frequency ν_{I2} is measured⁴. The goal is to express frequency ν_{F1} , which should be observed by F at the time of reception with use of the measured frequencies ν_{I0} and ν_{I2} and a computed correction. We express a ratio ν_{I2}/ν_{F1} as

$$\frac{\nu_{I2}}{\nu_{F1}} = \frac{1}{2} \frac{\nu_{I2}}{\nu_{I0}} + \Delta + \frac{1}{2} \tag{19}$$

where the first term on the right-hand side contains only the measured quantities and the correction Δ needs to be computed. For this correction we obtain (see appendix A for derivation)

$$\Delta = \delta_{I1} - \delta_{F1} + \frac{1}{c^2} \int_0^L \frac{\partial \mathbf{v} \cdot \mathbf{s}_l}{\partial t} dl \tag{20}$$

where the index $I1$ or $F1$ means the quantity is evaluated at the time $t = t_1$ in the initial or final end of the fibre, respectively, and we introduced $\delta \ll 1$ such that $dt/d\tau = 1 + \delta$. For the required accuracy of 10^{-18} the δ -terms in this equation depend only on the scalar potential and velocity at the end-points of the fibre and are studied in detail in [6]. In our case of space-time with the GCRS metric we obtain

$$\delta \equiv \frac{dt}{d\tau} - 1 = \frac{1}{c^2} \left(w + \frac{v^2}{2} \right) + O(c^{-4}). \tag{21}$$

The integral term of (20) is also evaluated at time $t = t_1$. This term can be expressed as

$$\int_0^L \frac{\partial \mathbf{v} \cdot \mathbf{s}_l}{\partial t} dl = \int_0^L \mathbf{a} \cdot \mathbf{s}_l dl + \frac{1}{2} (v_{F1}^2 - v_{I1}^2) \tag{22}$$

where $\mathbf{a} = \partial \mathbf{v} / \partial t = (\partial^2 x^i / \partial t^2) \mathbf{e}_i$ is the acceleration of points of the fibre in the GCRS frame and the second term was obtained using

$$\mathbf{v} \cdot \frac{\partial \mathbf{s}_l}{\partial t} = \delta_{ij} \frac{\partial x^i}{\partial t} \frac{\partial^2 x^j}{\partial t \partial l} = \frac{1}{2} \frac{\partial}{\partial l} \left(\delta_{ij} \frac{\partial x^i}{\partial t} \frac{\partial x^j}{\partial t} \right) = \frac{1}{2} \frac{\partial v^2}{\partial l}.$$

Inserting (21) and (22) into (20) we get another expression for Δ

$$\Delta = \frac{1}{c^2} (w_{I1} - w_{F1}) + \frac{1}{c^2} \int_0^L \mathbf{a} \cdot \mathbf{s}_l dl. \tag{23}$$

We can see that the terms depending on the velocity of the end-points cancel.

The largest contribution to acceleration \mathbf{a} is given by a centrifugal acceleration due to the Earth's rotation. To see the centrifugal acceleration explicitly we can express \mathbf{a} in terms

⁴ A generalisation of the formula in case there is a delay between the reception and the re-emission at observer F is possible.

of quantities related to the frame co-rotating together with the Earth as follows

$$\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) + 2\boldsymbol{\omega} \times \mathbf{v}_R + \boldsymbol{\varepsilon} \times \mathbf{x} + \mathbf{a}_R \tag{24}$$

where $\boldsymbol{\omega}$ is the angular velocity of the Earth's surface, \mathbf{v}_R is the velocity of the fibre points in the co-rotating frame, $\boldsymbol{\varepsilon}$ is the angular acceleration of the Earth's surface and \mathbf{a}_R is the acceleration of the fibre points in the co-rotating frame. In terms of co-rotating coordinates x_R^i and the corresponding basis \mathbf{e}_i^R the quantities can be expressed as $\mathbf{x} = x_R^i \mathbf{e}_i^R$, $\mathbf{v}_R = (\partial x_R^i / \partial t) \mathbf{e}_i^R$, $\mathbf{a}_R = (\partial^2 x_R^i / \partial t^2) \mathbf{e}_i^R$, $\boldsymbol{\omega} = \omega_R^i \mathbf{e}_i^R$ and $\boldsymbol{\varepsilon} = \varepsilon_R^i \mathbf{e}_i^R$ with $\varepsilon_R^i \equiv (\partial \omega_R^i / \partial t)$. The first term in (24) is the centrifugal acceleration which is followed by the Coriolis and Euler acceleration. Using the fact that

$$\begin{aligned} (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})) \cdot \mathbf{s}_l &= -(\boldsymbol{\omega} \times \mathbf{x}) \cdot (\boldsymbol{\omega} \times \mathbf{s}_l) \\ &= -\frac{1}{2} \frac{\partial (\boldsymbol{\omega} \times \mathbf{x})^2}{\partial t} \end{aligned} \tag{25}$$

we can integrate the centrifugal term explicitly and express it in terms of centrifugal potential $(\boldsymbol{\omega} \times \mathbf{x})^2 / 2 = (\omega R \sin \theta)^2 / 2$ with $R = \|\mathbf{x}\|$ and θ being an angle between the rotation axis and the position vector of the fibre end-point \mathbf{x} . Inserting (24) to (23) then gives

$$\begin{aligned} \Delta &= \frac{1}{c^2} (w_{I1} - w_{F1}) + \frac{1}{c^2} \left[\frac{(\boldsymbol{\omega} \times \mathbf{x}_I)^2}{2} - \frac{(\boldsymbol{\omega} \times \mathbf{x}_F)^2}{2} \right] \\ &+ \frac{1}{c^2} \int_0^L (\mathbf{a}_R + 2\boldsymbol{\omega} \times \mathbf{v}_R + \boldsymbol{\varepsilon} \times \mathbf{x}) \cdot \mathbf{s}_l dl \end{aligned} \tag{26}$$

where again all quantities are expressed at $t = t_1$.

To obtain the best accuracy for the endpoint terms of (26) given by sum of the scalar gravitational potential w and the centrifugal potential, the two contributions should not be computed separately but the procedure described in [6] should be followed.

Contributions to the integral term of (26) are given e.g. by the Earth's tides or by variations in the Earth's rotation. Considering the Earth's tides with a period of 12h and an amplitude of vertical Earth surface motion of approx. 30cm we obtain a maximal vertical velocity of the Earth's surface of approx. 0.05 mm s^{-1} as we already mentioned and a maximal vertical acceleration of approx. $6 \times 10^{-9} \text{ ms}^{-2}$. The maximal contribution of the Earth's tides is given by the Coriolis term in (26) for a fibre path following the equator. Its value in this maximising example is 8×10^{-20} per 1000 km of the fibre. The maximal contribution of the \mathbf{a}_R term in (26) due to the Earth's tides is around 3×10^{-20} per 1000 km of a fibre following a meridian near 45° latitude.

The angular velocity vector $\boldsymbol{\omega}$ precesses around a certain fixed axis in the co-rotating frame with a one-day period and with an evolving amplitude. We choose the basis of the co-rotating frame \mathbf{e}_R^i such that \mathbf{e}_R^z points in the direction of this axis. The angular acceleration components $\varepsilon_R^x, \varepsilon_R^y$ therefore oscillate with a one-day period and a maximal amplitude of approx. $5 \times 10^{-16} \text{ s}^{-2}$ mostly due to the changing direction of $\boldsymbol{\omega}$. This value was estimated based on the known evolution of the Earth's angular velocity vector in the Terrestrial

Table 2. Effects influencing the frequency transfer in optical fibres and sizes of the corresponding corrections in one-way and two-way frequency transfer formulas.

Effect	Correction
Difference of gravitational plus centrifugal potential at endpoints	$> 10^{-18}$
Variations in length and refractive index due to temperature changes (one-way only)	$\sim 10^{-13}$
Coriolis acceleration of the fibre due to velocity of the Earth's tides	8×10^{-20}
Euler acceleration of the fibre due to angular acceleration of the Earth's rotation	4×10^{-20}
Acceleration of the Earth's tides	3×10^{-20}

Note: The first correction depends on the fibre endpoints only. The remaining corrections depend on processes in the whole fibre and they are calculated per 1000 km of the fibre length. Values for the last three corrections are examples for specific fibre positions where the effect is maximised (see the text for details).

Reference Frame. The component ϵ_R^z is several orders below the x, y —components corresponding to the fact that the effect of changing magnitude of ω is much smaller than the effect of changing direction. The largest contribution of the Euler term in (26) would be observed for a fibre path following the meridian perpendicular to the instant direction of ϵ (neglecting the ϵ_R^z component). In this maximising example we obtain the contribution of 4×10^{-20} per 1000 km of the fibre.

The phenomena contributing to the frequency transfer corrections are summarised in table 2.

5. Required uncertainty of the input parameters

In this section we discuss what is the required uncertainty of the input parameters entering the formulas for time and frequency transfer in order to keep the uncertainty of the resulting corrections below the 1 ps target for time transfer and 10^{-18} target for frequency transfer.

5.1. Time transfer

The required uncertainty of the length of the fibre in the case of one-way time transfer would be 0.2 mm to achieve a 1 ps uncertainty of the propagation time. For a 1000 km long fibre the required uncertainty of the effective refractive index would be 3×10^{-10} . Since these uncertainties are difficult to achieve in practice, e.g. because of unknown temperature variations, the two-way transfer is used. Therefore, we focus on the evaluation of the two-way time transfer formula (10) only.

For an evaluation of (10) we use its form given by (14). First we investigate uncertainty of the Sagnac term due to uncertain position of the fibre and we look for a requirement for the position uncertainty leading to the Sagnac term within the 1 ps uncertainty limit. We consider that the fibre position at a certain coordinate time is shifted by a vector $\xi(l)$, i.e. the new position vector is $\mathbf{x}(l) + \xi(l)$. The corresponding change of the Sagnac term can be obtained by inserting this new position vector to the formula (13), where $\mathbf{s}_l = d\mathbf{x}/dl$. Expanding this formula and integrating by parts we obtain

$$2\omega\Delta A = \omega \cdot (\mathbf{x}_F \times \xi_F) - \omega \cdot (\mathbf{x}_I \times \xi_I) + 2 \int_0^L \omega \cdot \left(\xi \times \left(\mathbf{s}_l + \frac{1}{2} \frac{d\xi}{dl} \right) \right) dl \quad (27)$$

where \mathbf{x}_I, ξ_I are the position vector and shift vector at the initial end of the fibre and similarly \mathbf{x}_F, ξ_F are the quantities at the final end. The change of the Sagnac term in the time transfer correction (10) is given as $2\omega\Delta A/c^2$.

First we look to the contribution of the endpoint terms of the formula (27). These terms correspond to a change of the Sagnac area given by a shift of the lines connecting the endpoints of the fibre with the Earth's center and they are maximised when the endpoints of the fibre lie on the equator and when the shift vectors point along the equator. In this case we have e.g. $\omega \cdot (\mathbf{x}_F \times \xi_F) = \omega R_E \xi_F$ with R_E being the Earth's radius and $\xi_F = \|\xi_F\|$ is the Euclidean magnitude of ξ_F . We get $\omega R_E \xi_F/c^2 = 1\text{ps}$ for $\xi_F \approx 200$ m, i.e. a shift of an endpoint of the fibre by not more than approx. 200 m can cause a shift in time transfer by not more than 1 ps.

Next we look to the contribution of the integral term of (27). This term corresponds to the Sagnac area in between the original and shifted fibre paths. If we assume that the shift ξ does not change the rest length of the fibre, we can derive the following upper bound

$$\left| 2 \int_0^L \omega \cdot \left(\xi \times \left(\mathbf{s}_l + \frac{1}{2} \frac{d\xi}{dl} \right) \right) dl \right| \leq 2\omega \bar{\xi} L \quad (28)$$

where $\bar{\xi}$ is an average shift magnitude defined as

$$\bar{\xi} = \frac{1}{L} \int_0^L \|\xi\| dl. \quad (29)$$

The corresponding upper bound for change in the correction (10) is therefore $2\omega \bar{\xi} L/c^2$. It equals to 1 ps for $\bar{\xi} \approx 600 \text{ km}^2 \times L^{-1}$. For an example of a 1000 km long fibre it means that an average uncertainty of the fibre's position of better than approx. 600 m is sufficient for a 1 ps uncertainty in the time transfer.

Including a fibre's expansion into $\xi(l)$ adds a minor correction to the bound (28) for expectable fibre length uncertainties.

Next we check the effect of changing ω for the time transfer. The condition $2\Delta\omega A/c^2 = 1$ ps for an equatorial fibre where A is maximised leads to $\Delta\omega/\omega \approx 200 \text{ m} \times L^{-1}$. For a 1000 km long fibre it corresponds to a relative uncertainty of an order of 10^{-4} in ω . The fluctuations of the Earth's angular velocity are much smaller than this, so the Earth's angular velocity is not an issue from the uncertainty point of view.

The second term of (14) can be expressed as a product $L\bar{v}_{Rl}$ where we defined an average tangent velocity of the fibre in the co-rotating frame as $\bar{v}_{Rl} \equiv \frac{1}{L} \int_0^L \mathbf{v}_R \cdot \mathbf{s}_l dl$. The 1 ps change in time transfer correction due to a variation in \bar{v}_{Rl} occurs if $L\Delta\bar{v}_{Rl}/c^2 = 1$ ps, i.e. the maximal allowed uncertainty of \bar{v}_{Rl} for 1 ps time transfer is $\Delta\bar{v}_{Rl} = 9 \times 10^4 \text{ m}^2/\text{s} \times L^{-1}$, which for a 1000 km long fibre gives 0.09 m s^{-1} . If the fibre is fixed to the Earth's surface the magnitude of \mathbf{v}_R itself is usually much smaller than this. For example, the maximal velocity caused by the Earth's tides is around 0.05 mm s^{-1} , as already mentioned.

Table 3. Input parameters and their maximal uncertainties sufficient for 1 ps uncertainty in time transfer.

Parameter	Uncertainty
Fibre length (1-way only)	0.2 mm
Refractive index (1-way only)	3×10^{-10}
Fibre endpoints position	200 m
Fibre inner points position	600 m
Fibre velocity in co-rotating frame	9 cm s^{-1}
Earth's angular velocity	$\sim 0.01\%$ (relative)
Gravitational plus centrifugal potential (1-way only)	$\sim 30\%$ (relative)

Note: The values were obtained for situations where the sensitivity of a correction to a parameter is maximised and they are calculated for a 1000 km long fibre (see the text for further details and scaling of the uncertainties with the fibre length).

Table 4. Input parameters and their maximal uncertainties sufficient for 10^{-18} relative uncertainty in frequency transfer.

Parameter	Uncertainty
Time derivative of the fibre temperature (change in length and refractive index; 1-way only)	$3 \times 10^{-11} \text{ K s}^{-1}$
Fibre velocity in a co-rotating frame	0.6 mm s^{-1}
Fibre acceleration in a co-rotating frame	$9 \times 10^{-8} \text{ ms}^{-2}$
Fibre position	> Earth's radius
Earth's angular velocity	> 100% (relative)
Earth's angular acceleration	> 100% (relative)

Note: The values were obtained for situations where the sensitivity of a correction to a parameter is maximised and they are calculated for a 1000 km long fibre (see the text for further details and scaling of the uncertainties with the fibre length). Parameters and uncertainties related to the difference of gravitational plus centrifugal potential at the fibre endpoints are not discussed here.

A variation of L in the product $L\bar{v}_{Rt}$ leads to the 1 ps shift in time transfer if $\Delta L\bar{v}_{Rt}/c^2 = 1$ ps. For expectable values of \bar{v}_{Rt} this leads to values of ΔL which are much larger than the usual uncertainty of length measurement. Therefore, the uncertainty about the fibre length is not an issue in this term.

The input parameters for time transfer and their required uncertainties are summarised in table 3.

5.2. Frequency transfer

The required uncertainty of the time derivative of the temperature of the fibre would be $3 \times 10^{-11} \text{ K s}^{-1}$ to achieve the 10^{-18} uncertainty in a one-way frequency transfer for a 1000 km long fibre. Since this is difficult to achieve in practice the two-way transfer is used. Therefore, for the frequency transfer we also focus on the analysis of the two-way correction which is given by (26). The end-point terms given by the sum of the scalar gravitational potential and centrifugal potential need to be known with an uncertainty of $0.09 \text{ m}^2 \text{ s}^{-2}$ in order to achieve the 10^{-18} uncertainty of Δ (see [6] for a detailed discussion). We focus here on the integral term in the second line of (26), which contains the contributions of the acceleration of the fibre in the co-rotating frame and of the Coriolis and Euler acceleration.

We define an average tangent acceleration of the fibre in the co-rotating frame as $\bar{a}_R \equiv \frac{1}{L} \int_0^L \mathbf{a}_R \cdot \mathbf{s}_l \, dl$. This quantity should be known with an uncertainty better than $\Delta\bar{a}_R = 0.09 \text{ m}^2 \text{ s}^{-2} \times L^{-1}$ in order to achieve the 10^{-18} uncertainty in (26). For a 1000 km long fibre it gives $9 \times 10^{-8} \text{ ms}^{-2}$. If we consider the Earth's tides as the source of the fibre acceleration in the co-rotating frame then its maximal value would be around $6 \times 10^{-9} \text{ ms}^{-2}$.

Now we consider that the velocity of the fibre in the co-rotating frame is changed from a value \mathbf{v}_R to a value $\mathbf{v}_R + \Delta\mathbf{v}_R$. The corresponding change of the Coriolis term in (26) is then given as

$$\int_0^L (2\boldsymbol{\omega} \times \Delta\mathbf{v}_R) \cdot \mathbf{s}_l \, dl . \tag{30}$$

For this change the following upper bound can be derived

$$\left| \int_0^L (2\boldsymbol{\omega} \times \Delta\mathbf{v}_R) \cdot \mathbf{s}_l \, dl \right| \leq 2\omega\Delta\bar{v}_R L \tag{31}$$

where $\Delta\bar{v}_R$ is an average magnitude of $\Delta\mathbf{v}_R$ defined as

$$\Delta\bar{v}_R = \frac{1}{L} \int_0^L \|\Delta\mathbf{v}_R\| \, dl . \tag{32}$$

The corresponding upper bound for change in the correction (26) is therefore $2\omega\Delta\bar{v}_R L/c^2$. It equals to 10^{-18} for $\Delta\bar{v}_R \approx 600 \text{ m}^2 \text{ s}^{-1} \times L^{-1}$. For an example of a 1000 km long fibre it means that the average uncertainty of the fibre velocity in a co-rotating frame better than 0.6 mm s^{-1} is sufficient for 10^{-18} uncertainty in frequency transfer. If we consider e.g. the Earth's tides as a source of motion of the fibre the maximal velocity of the fibre in a co-rotating frame would be around 0.05 mm s^{-1} .

The uncertainty requirement for the Earth's angular velocity in the Coriolis term is not an issue since the expected value of \mathbf{v}_R is very small.

For a change of the Euler term in (26) due to a variation $\Delta\boldsymbol{\epsilon}$ in the angular acceleration vector we obtain the following upper bound

$$\left| \int_0^L (\Delta\boldsymbol{\epsilon} \times \mathbf{x}) \cdot \mathbf{s}_l \, dl \right| \leq R_m \Delta\epsilon L \tag{33}$$

where R_m is the maximal value of $\|\mathbf{x}\|$ approximately given by the Earth's radius and $\Delta\epsilon = \|\Delta\boldsymbol{\epsilon}\|$. The corresponding upper bound in the correction (26) is then $R_m \Delta\epsilon L/c^2$ and it equals to 10^{-18} for $\Delta\epsilon = 1.4 \times 10^{-8} \text{ ms}^{-2} \times L^{-1}$. For example, an uncertainty of $1.4 \times 10^{-14} \text{ s}^{-2}$ in angular acceleration is sufficient for 10^{-18} uncertainty in frequency transfer for a 1000 km long fibre. But, the value of ϵ itself usually does not exceed $5 \times 10^{-16} \text{ s}^{-2}$, as we already mentioned.

The input parameters for frequency transfer and their required uncertainties are summarised in table 4.

6. Numerical calculation of corrections for the NPL-SYRTE and PTB-SYRTE fibre links

In this section we evaluate the relativistic corrections for time and frequency transfer in real fibre links connecting SYRTE

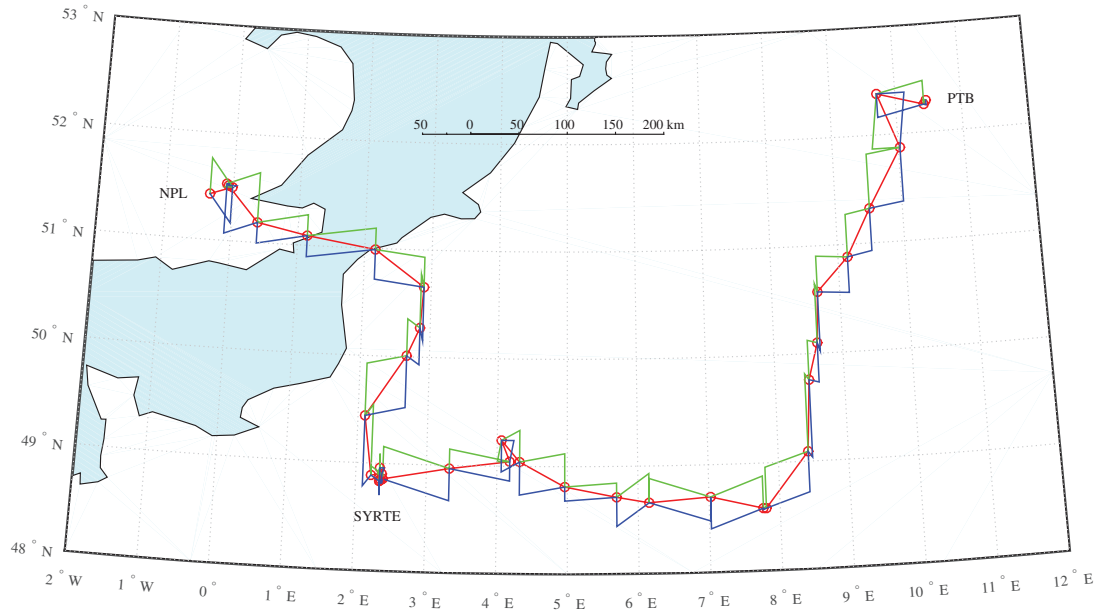


Figure 3. Paths for the NPL-SYRTE and the PTB-SYRTE fibres. Positions of the shelters are depicted as circles. The direct path in between the shelters (red in online version), maximum path (blue in online version) and minimum path (green in online version) are shown.

(Paris, France) with NPL (Teddington, United Kingdom) and with PTB (Braunschweig, Germany) (see figure 3). These fibres are part of the REFIMEVE+ project⁵, which aims to build a metrological fibre network in Europe.

We focus on the two-way time and frequency transfer since the uncertainty of the leading c^{-1} terms in one-way transfer coming e.g. from an uncertainty of fibre length would lead to an inaccuracy that would completely surpass the relativistic corrections.

For the two-way time transfer we compute the main contribution given by the Sagnac correction (10). In the two-way frequency transfer formula (20) the main contribution is given by the δ -terms containing the scalar potentials and velocities at the endpoints of the fibre. As we already mentioned the evaluation of this term requires a high accuracy knowledge of the gravitational potentials and it is a subject of a separate research.

Here we focus on the second term which comes from the time derivative of the Sagnac term. This term is the largest contribution coming from the processes in the fibre (not just from the state of the endpoints).

6.1. Discretized equations for the Sagnac correction

For the numerical computation we suppose that we know the positions of certain points of the fibre, and in between these points we consider the fibre to be a straight line in the Euclidean sense in the GCRS coordinates. We suppose that we know $N + 1$ points of the fibre and we index the points by index in brackets, i.e. $(i) = 1, \dots, N + 1$. We have defined an initial point of the fibre I and endpoint of the fibre F and we suppose that the index (i) is growing from I to F with $(i) = 1$ at I and $(i) = N + 1$ at F . We need to express the formulas for corrections in terms of coordinates of the known points and their time derivatives. We denote the spatial GCRS coordinates

of the i th fibre point by $x_{(i)}^a(t) = (x_{(i)}(t), y_{(i)}(t), z_{(i)}(t))$. Using (6) the Sagnac correction term can be expressed as

$$\int_0^L \mathbf{v} \cdot \mathbf{s}_l dl = \int_0^L \delta_{ab} \frac{\partial x^a}{\partial t} \frac{\partial x^b}{\partial l} dl. \quad (34)$$

We consider the motion of straight fibre segments as rigid body motion, i.e. the velocity $\partial x^a / \partial t$ changes linearly with the Euclidean length l_E from one endpoint of the segment to another. In this case we obtain

$$\int_0^L \mathbf{v} \cdot \mathbf{s}_l dl = \frac{1}{2} \sum_{i=1}^N \delta_{ab} \left(\frac{dx_{(i)}^a}{dt} + \frac{dx_{(i+1)}^a}{dt} \right) (x_{(i+1)}^b - x_{(i)}^b). \quad (35)$$

In frequency transfer a time derivative of (34) occurs. For numerical computation we can use a formula which we obtain directly by time derivative of the right-hand side of (35).

If the effect of fibre velocity \mathbf{v}_R in the frame co-rotating with the Earth is negligible (this velocity can appear e.g. due to the Earth's tides) then we can use the formula (14) to express the Sagnac correction in a way which contains the Earth's rotation explicitly. The Sagnac area A can be expressed using the formula (13). If we denote $\mathbf{x}_{(i)}$ a position vector connecting the centre of the GCRS system with the i th fibre point the Sagnac term can be expressed as

$$2\omega A = \sum_{i=1}^N \boldsymbol{\omega} \cdot (\mathbf{x}_{(i)} \times \mathbf{x}_{(i+1)}) = \sum_{i=1}^N \epsilon_{abc} \omega^a x_{(i)}^b x_{(i+1)}^c. \quad (36)$$

The time derivative needed for frequency transfer can be obtained directly by differentiating the formula (36) with respect to time.

6.2. Fibre routing

The path of the fibre between PTB and SYRTE, and between NPL and SYRTE, is poorly known. However, the fibre is going

⁵ www.refimeve.fr/index.php/en/.

Table 5. Relativistic corrections for two-way time transfer in the case of the PTB-SYRTE and NPL-SYRTE fibre links.

Fibre link	Length/km	Correction/ps
PTB-SYRTE	1401	3976 ± 27
NPL-SYRTE	813	1214 ± 6

Note: The only contribution relevant for the 1 ps accuracy comes from the Sagnac term. The \pm values indicate the estimated maxima and minima of the correction as described in the text.

through several shelters: the positions of these shelters have been measured with GPS, and the length of the fibre between these shelters has been measured with a laser passing through the fibre. There are 23 known shelters along the PTB-SYRTE path, and 15 along the NPL-SYRTE path. From these data we build three different fibre paths:

- (i) the direct path: this is obtained by directly connecting the shelters with straight lines. This path largely underestimates the total fibre length.
- (ii) the maximum path: for this path we fix the fibre lengths between each shelter to the measured values. An infinite number of paths can be followed by the fibre between two given shelters, giving different values for the Sagnac effect; we choose one that approximates the maximum of the Sagnac effect.
- (iii) the minimum path: as for the maximum path, the fibre lengths between each shelters are fixed to the measured values. Then we choose a path that approximates the minimum of the Sagnac effect.

These three paths are shown on a map in figure 3.

6.3. Evaluation of the relativistic corrections for NPL-SYRTE and PTB-SYRTE fibre links

6.3.1. Two-way time transfer. The values obtained for the two-way time transfer Sagnac effect (10) can be found in table 5. The model used is accurate to better than 1 ps; however the uncertainty in the fibre position leads to an uncertainty in the computed correction bigger than 1 ps. We estimate the uncertainty on the Sagnac effect due to the poorly known fibre path by calculating the Sagnac effect for the three paths described in the previous paragraph. The differences of the Sagnac effect between the maximum and the minimum paths are:

- PTB-SYRTE paths: 53 ps
- NPL-SYRTE paths: 12 ps

There are also variations in the Sagnac term due to the Earth's tides and due to changes in the angular velocity vector of the Earth. All these variations, however, lead to corrections of order or less than 5 fs. Tides also imply a residual velocity with respect to the co-rotating frame, which leads to a correction which is much less than 1 ps.

6.3.2. Two-way frequency transfer. In the time derivative of the Sagnac correction in (20) there are contributions coming from the Earth's tides and from the variation in the angular velocity vector of the Earth. All these contributions vary and have amplitudes of order 10^{-19} or less, which corresponds to a variation of around 5 fs over 12 h in the time transfer.

These terms are not studied in detail here as they are below the accuracy goal. However, it would be interesting to model them accurately as they are just one order of magnitude below the limit of the actual clock stability. By integrating the clock's frequency comparison for a long time it could be detectable.

7. Conclusion

In this work we presented a systematic relativistic description of the propagation of a signal in optical fibres. We derived a general differential equation governing the signal propagation and we investigated its solution up to terms of order c^{-3} . This is sufficient for an accuracy of 1 ps for time transfer formulas and for a relative accuracy of 10^{-18} for frequency transfer formulas. This accuracy corresponds to the requirements of nowadays and near future optical clock applications.

Formulas for both one-way and two-way time and frequency transfer were derived. The largest relativistic correction for one-way and two-way time transfer is the so-called Sagnac effect, which strongly depends on the fibre positioning on the Earth's surface and its magnitude usually does not exceed an order of nanoseconds. The next significant correction for one-way transfer takes the gravitational field into account and depends on the gravity potential along the fibre. This correction usually does not exceed an order of picoseconds.

For frequency transfer there is the well-known gravitational red shift, which depends only on the state of the endpoints of the fibre. Moreover, there are other effects which depend on the processes in the fibre itself. For two-way transfer the largest effects are due to the change of the fibre position with time and due to thermal variations. The contribution of these effects was estimated to be smaller than 10^{-18} .

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Appendix A. Theory

Appendix A.1. Constrained relativistic propagation equation of signal in a medium

We use the following conventions. Small Greek indices go from 0 to 3, small Latin indices go from 1 to 3, and the metric signature is $- + + +$.

First we will derive a condition governing signal propagation in an optical fibre. If we denote U^α the four-velocity field of the fibre's motion and k^α a tangent vector to a trajectory of the signal propagating in the fibre the condition (1) can

be written in terms of the components of vector k^α in the rest frame of the observer U^α as (we drop the index eff)

$$c \frac{\sqrt{\delta_{ab} k^a k^b}}{k^0} = \frac{c}{n}. \tag{A.1}$$

The terms containing the components of vector k^α can be written in covariant form as follows

$$k^0 = -U_\alpha k^\alpha \tag{A.2}$$

$$\delta_{ab} k^a k^b = (g_{\alpha\beta} + U_\alpha U_\beta) k^\alpha k^\beta \tag{A.3}$$

where $g_{\alpha\beta}$ is the spacetime metric and the expression for $\delta_{ab} k^a k^b$ was obtained as a square of the orthogonal projection of k^α into the normal space of U^α , which is given as $(\delta_\beta^\alpha + U^\alpha U_\beta) k^\beta$. Inserting (A.2) and (A.3) into (A.1) we obtain the following condition for the vector k^α

$$n \sqrt{(g_{\alpha\beta} + U_\alpha U_\beta) k^\alpha k^\beta} = -U_\alpha k^\alpha. \tag{A.4}$$

This condition can also be written as $\gamma_{\alpha\beta} k^\alpha k^\beta = 0$ with $\gamma_{\alpha\beta}$ being defined as

$$\gamma_{\alpha\beta} \equiv g_{\alpha\beta} + \left(1 - \frac{1}{n^2}\right) U_\alpha U_\beta. \tag{A.5}$$

This means that the signal propagates along null lines of a modified metric $\gamma_{\alpha\beta}$. Besides exchanging a refractive index for the effective refractive index, this is exactly the metric introduced by Gordon in [17].

Now we are going to derive an equation for signal propagation in optical fibre based on the condition (A.4).

We consider a spacetime with time coordinate t . The fibre is considered to be a one-dimensional object, i.e. its trajectory is a 2D surface in spacetime. The fibre is parametrised by two parameters—the coordinate time t and a parameter λ , which has a unique value for a given fibre element. The parameter λ has a range $\lambda \in [\lambda_I, \lambda_F]$ and is growing from an initial point of the fibre, which we denote I to the final point of the fibre, which we denote F . If x^α are coordinates in the spacetime the fibre worldsurface is given parametrically as

$$x^\alpha = x^\alpha(t, \lambda). \tag{A.6}$$

For a fixed value of λ this equation gives a trajectory of a corresponding fibre element. Therefore, the four-velocity of the fibre U^α is proportional to a partial derivative of $x^\alpha(t, \lambda)$ with respect to time. We have

$$U^\alpha = U^\beta \partial_\beta t \frac{\partial x^\alpha}{\partial t} \tag{A.7}$$

where the proportionality factor can be expressed using a normalisation condition $U^\alpha U_\alpha = -1$ as

$$U^\alpha \partial_\alpha t = \left(-g_{\alpha\beta} \frac{\partial x^\alpha}{\partial t} \frac{\partial x^\beta}{\partial t} \right)^{-\frac{1}{2}}. \tag{A.8}$$

The parameters t, λ can be seen as coordinates on the worldsurface of the fibre. The signal trajectory can be therefore described by functions $t(\sigma), \lambda(\sigma)$, where σ is a parametrisation of the signal trajectory. In coordinates x^α the trajectory

reads $x^\alpha(t(\sigma), \lambda(\sigma))$. It will be convenient to choose the parameter $\sigma = \pm \lambda$, where $+$ is for the propagation from I to F , i.e. with increasing λ , and $-$ is for the propagation from F to I , i.e. with decreasing λ . In this case the tangent to the trajectory can be expressed as

$$k^\alpha = \frac{dx^\alpha}{d\sigma} = \pm \frac{\partial x^\alpha}{\partial t} \frac{dt}{d\lambda} \pm \frac{\partial x^\alpha}{\partial \lambda}. \tag{A.9}$$

Inserting (A.9) to (A.4) with the use of (A.7) and with the use of the fact that the projection of $\partial x^\alpha / \partial t$ to normal space of U^α is zero we obtain the following equation

$$\frac{dt}{d\lambda} = U^\alpha \partial_\alpha t \left(\pm n \frac{\partial t}{\partial \lambda} + U_\beta \frac{\partial x^\beta}{\partial \lambda} \right) \tag{A.10}$$

where we defined a function

$$l_t(t, \lambda) = \int_{\lambda_I}^\lambda \sqrt{(g_{\alpha\beta} + U_\alpha U_\beta) \frac{\partial x^\alpha}{\partial \lambda} \frac{\partial x^\beta}{\partial \lambda}} d\lambda \tag{A.11}$$

which is a rest length of the fibre at a time t between elements λ_I and λ .

The equation (A.10) is a differential equation for the function $t(\lambda)$, which describes the trajectory of the signal in the fibre in terms of coordinate time t , at which the signal reaches an element λ of the fibre. The sign $+$ in the equation relates to the signal propagation from I to F , i.e. along the orientation of the λ coordinate and the sign $-$ relates to propagation from F to I .

Appendix A.2. Propagation equation for optical fibre on the Earth's surface

Now we are going to express the right-hand side of the equation (A.10) for a particular case of a fibre located on the Earth's surface. We can do this using the formulas (A.7) and (A.8), where x^α are chosen to be the GCRS coordinates (ct, x, y, z) with metric components $g_{\alpha\beta}$ given by (2)–(4). The derivatives $\partial x^\alpha / \partial t$ and $\partial x^\alpha / \partial \lambda$ which occur in (A.7), (A.8) and (A.10) can be expressed in terms of the velocity vector field and tangent vector field of the fibre defined by (6).

Moreover, a Euclidean vector corresponding to the vector gravitational potential w_i can be defined as

$$\mathbf{w} = w_i \delta^{ij} \mathbf{e}_j. \tag{A.12}$$

Expanding the right-hand side of the equation (A.10) in powers of c^{-1} we obtain

$$\begin{aligned} \frac{dt_\pm}{d\lambda} = & \pm \frac{n}{c} \frac{\partial t}{\partial \lambda} + \frac{1}{c^2} \mathbf{v} \cdot \mathbf{s}_\lambda \pm \frac{1}{c^3} (w + v^2/2) n \frac{\partial t}{\partial \lambda} \\ & + \frac{1}{c^4} ((4w + v^2) \mathbf{v} \cdot \mathbf{s}_\lambda - 4\mathbf{w} \cdot \mathbf{s}_\lambda) + O(c^{-5}) \end{aligned} \tag{A.13}$$

where we denoted $v^2 = \mathbf{v} \cdot \mathbf{v}$ and we added the \pm sign also to the left-hand side of the equation to distinguish the solutions for the propagation in different directions.

The first term on the right-hand side of (A.13) corresponds to the Newtonian limit. The second term corresponds to the Sagnac correction. The third term corresponds to a part of the

Shapiro correction. Another part would appear if we used a Euclidean length of the fibre in the GCRS coordinates as the parameter λ (see the discussion below). The fourth term contains the geodetic and Lense–Thirring effect.

The parameter λ in (A.13) is an arbitrary parameter satisfying the condition that its value is unique (not changing in time) for a given fibre element. However, some natural choices can be done which are convenient for practical purposes. One possibility for such a parameter is the rest length l_t itself at a specific time t_1 . The fibre can expand e.g. due to temperature changes and therefore l_t must be fixed at a certain time to fulfill the uniqueness condition. We denote this parameter l , i.e. $l(\lambda) = l_t(t_1, \lambda)$. If we consider the thermal expansion we have

$$\frac{\partial l_t}{\partial l} = 1 + \alpha(T(t, l) - T(t_1, l)) \tag{A.14}$$

where $T(t, l)$ is the temperature of the fibre at a given time and position and α is its linear thermal expansion coefficient.

Another choice of the parameter could be a Euclidean length of the fibre at a time t_1 . The Euclidean length at a time t between the fibre's elements λ_t and λ is defined as

$$l_{E_t}(t, \lambda) = \int_{\lambda_t}^{\lambda} \sqrt{\delta_{ab} \frac{\partial x^a}{\partial \lambda} \frac{\partial x^b}{\partial \lambda}} d\lambda \tag{A.15}$$

We denote the Euclidean parameter of the fibre l_E , i.e. $l_E(\lambda) = l_{E_t}(t_1, \lambda)$. We obtain

$$\frac{\partial l}{\partial l_E} = 1 + \frac{1}{c^2} \left(w + \frac{(\mathbf{v} \cdot \mathbf{s})^2}{2} \right) + O(c^{-4}) \tag{A.16}$$

with \mathbf{s} being a unit tangent to the fibre, i.e. $\mathbf{s} \cdot \mathbf{s} = 1$.

Choosing l_E as the parameter of the fibre then leads to the terms in (A.13) given by $\partial l_t / \partial l_E = (\partial l_t / \partial l) \cdot (\partial l / \partial l_E)$ with the derivatives on the right-hand side given by (A.14) and (A.16).

Appendix A.3. Solving the equation for signal propagation

Now we are going to solve the differential equation (A.13). We denote the right-hand side of (A.13) by $\Omega_{\pm}(t, \lambda)$ with the $+$ sign referring to propagation from I to F , and the $-$ sign referring to propagation in opposite direction. Then the equation (A.13) can be written as

$$\frac{dt_{\pm}}{d\lambda} = \Omega_{\pm}(t, \lambda) \tag{A.17}$$

If Ω_{\pm} does not depend on t we can integrate the equation directly. However, Ω_{\pm} depends on time t , because of e.g. thermal expansion, the Earth's tides or variations in the instantaneous rotation vector of the Earth. Nevertheless, these time dependencies are very slow, and we will see that the value of Ω_{\pm} does not change significantly during the propagation time of the signal from one end of the fibre to the other, which is around 5 ms for a 1000 km fibre with index $n = 1.5$.

We consider a signal emitted from observer I at time t_0 , reflected from observer F at time t_1 and received back by

observer I at time t_2 , as depicted in figure 1. We use l as the parameter of the fibre, which was defined as $l(\lambda) = l_t(t_1, \lambda)$. The range of the parameter is $l \in [0, L]$ with $L = l(\lambda_F)$ being the total rest length of the fibre at time t_1 .

The formal solution of the equation (A.17) can be written in a form

$$t_{\pm}(l) = t_1 + \int_L^l \Omega_{\pm}(t_{\pm}(l), l) dl \tag{A.18}$$

The formula (A.18) can be further processed by iterations. In the first iteration we set $t_{\pm}(l) = t_1$ on the right-hand side of (A.18). Therefore, we obtain

$$t_{\pm}^{[1]}(l) = t_1 + \int_L^l \Omega_{\pm}(t_1, l) dl \tag{A.19}$$

where we marked the iteration number in square brackets. The second iteration then gives

$$t_{\pm}^{[2]}(l) = t_1 + \int_L^l \Omega_{\pm} \left(t_1 + \int_L^{l'} \Omega_{\pm}(t_1, l') dl', l \right) dl \tag{A.20}$$

The order of deviation of the exact solution from the second iteration is given by

$$t_{\pm}(l) = t_{\pm}^{[2]}(l) + O \left[\left(\frac{\partial \Omega_{\pm}}{\partial t} \right)^2 \delta l^2 \delta t(l) \right] \tag{A.21}$$

where $\delta l = l - L$ and $\delta t(l)$ is the propagation time between l and L . This deviation turns out to be negligible for our level of accuracy, e.g. for a 1000 km long fibre, and for the effects caused by temperature variations it gives 10^{-28} s. Therefore, we use the formula (A.20) for $t_{\pm}(l)$.

Now we expand $\Omega_{\pm}(t, l)$ in (A.20) in time around t_1 . We obtain

$$t_{\pm}(l) = t_1 + t_{\pm}^{(0)}(l) + t_{\pm}^{(1)}(l) + O \left[\frac{\partial^2 \Omega_{\pm}}{\partial t^2} \delta l \delta t(l)^2 \right] \tag{A.22}$$

where

$$t_{\pm}^{(0)}(l) = \int_L^l \Omega_{\pm}(t_1, l) dl \tag{A.23}$$

$$t_{\pm}^{(1)}(l) = \int_L^l \frac{\partial \Omega_{\pm}}{\partial t}(t_1, l) \left[\int_L^{l'} \Omega_{\pm}(t_1, l') dl' \right] dl \tag{A.24}$$

The term of (A.24) in square brackets can be expressed using the Newtonian limit of Ω_{\pm} given by the c^{-1} term of (A.13), where we consider the refractive index to be a constant, which we denote n_0 . Thus we get

$$t_{\pm}^{(1)}(l) = \pm \frac{n_0}{c} \int_L^l \frac{\partial \Omega_{\pm}}{\partial t}(t_1, l) (l - L) dl \tag{A.25}$$

Now we can check the magnitude of the contribution (A.25). For our estimation we use the Newtonian limit of Ω_{\pm} to calculate $\partial \Omega_{\pm} / \partial t$ again. We obtain

$$\frac{\partial \Omega_{\pm}}{\partial t}(t_1, l) \approx \pm \frac{\partial}{\partial t} \left(\frac{n}{c} \frac{dl_t}{dl} \right) = \pm \frac{1}{c} \left(\frac{\partial n}{\partial t} + n \alpha \frac{\partial T}{\partial t} \right) \tag{A.26}$$

where we considered that the expansion of the fibre is caused by a change in temperature and we used the formula (A.14). The order of (A.25) is then

$$t_{\pm}^{(1)}(l) \sim \frac{1}{2} \frac{n_0}{c^2} \left(\frac{\partial n}{\partial t} + n\alpha \frac{\partial T}{\partial t} \right) (l - L)^2. \quad (\text{A.27})$$

For our estimation we consider a change of n due to changing temperature, i.e. $\partial n/\partial t = (\partial n/\partial T) \cdot (\partial T/\partial t)$. Based on the values given by (17) and (18) we estimate:

$$t_{\pm}^{(1)} \sim 2 \times 10^{-11} \text{ s}^{-1} \times n_0 \left(\frac{L}{c} \right)^2 \sim 0.3 \text{ fs} \quad (\text{A.28})$$

for a fibre with $L \sim 1000$ km and $n_0 \sim 1.5$. Therefore, a maximum variation of 0.6 fs occurs in 12 h, corresponding to a change in the relative frequency comparison of $\sim 1.5 \times 10^{-20}$, below the required accuracy.

In the following we will not consider the contribution $t_{\pm}^{(1)}$. Finally, we get the solution to the required accuracy

$$t_{\pm}(l) = t_1 + \int_L^l \Omega_{\pm}(t_1, l) dl. \quad (\text{A.29})$$

Appendix A.4. Time transfer

In time transfer the coordinate times of the signal propagation between the fibre endpoints are needed. We denote Δt_+ the coordinate time of the signal propagation from I to F and Δt_- the coordinate time of the signal propagation from F to I , i.e.

$$\Delta t_+ = t_1 - t_0 = t_1 - t_+(0), \quad (\text{A.30})$$

$$\Delta t_- = t_2 - t_1 = t_-(0) - t_1. \quad (\text{A.31})$$

Using the formula (A.28) we obtain

$$\Delta t_{\pm} = \pm \int_0^L \Omega_{\pm}(t_1, l) dl. \quad (\text{A.32})$$

Inserting the Ω_{\pm} given by (A.13) into (A.32) up to the c^{-3} order we obtain the formula (7).

Appendix A.5. Frequency transfer

For one-way frequency transfer we start with the formula (15) describing the ratio of proper frequencies of a signal as observed during its emission at one endpoint of the fibre and reception at the opposite endpoint.

The derivatives dt_e/dt_e and dt_r/dt_r in (15) are discussed in the main part of the text.

The derivative dt_r/dt_e in (15) can be evaluated using the formula (A.29). For propagation from I to F we have $dt_{r+}/dt_e = (\partial t_+(0)/\partial t_1)^{-1}$ and for propagation from F to I we have $dt_{r-}/dt_e = \partial t_-(0)/\partial t_1$. Therefore, we get

$$\frac{dt_{r\pm}}{dt_e} = 1 \pm \int_0^L \frac{\partial \Omega_{\pm}}{\partial t}(t_1, l) dl \quad (\text{A.33})$$

where we used $(1 - \varepsilon)^{-1} \approx 1 + \varepsilon$ for the + direction and therefore we neglected terms of order ε^2 , i.e. of order

$(\partial \Omega_{\pm}/\partial t)^2 L^2 \sim 10^{-38} \text{ m}^{-2} \times L^2$ in the case of thermal variations of Ω_{\pm} .

In the formula (A.33) we also neglected the terms coming from the higher order variations of Ω_{\pm} . In particular, we have seen that the first order term of expansion (A.22) given by (A.27) leads to a correction in the frequency transfer:

$$\frac{dt_{r\pm}^{(1)}}{dt_e} \sim 9 \times 10^{-16} \text{ s}^{-2} \times n_0 \left(\frac{L}{c} \right)^2 \sim 1.5 \times 10^{-20} \quad (\text{A.34})$$

for a fibre with $L \sim 1000$ km and $n_0 \sim 1.5$.

Inserting the Ω_{\pm} given by (A.13) into (A.33) up to the c^{-2} order and with the use of (A.26) we obtain the formula (16).

The two-way frequency transfer is described by the formula (19). This formula can be understood as the definition of the correction Δ which needs to be computed. The Δ can be expressed from (19) and further processed using the formula (15) for the frequency ratios with dt_r/dt_e given by (A.33) and dt/dt_r calculated from the metric.

We write $dt/dt_r = 1 + \delta$ with $\delta \ll 1$ (on the Earth's surface $\delta \sim 10^{-9}$). Therefore, we neglect terms where δ multiplies the integrals of $\partial \Omega_{\pm}/\partial t$ which, for the case of thermal variations of Ω_{\pm} , are of order $L(\partial \Omega_{\pm}/\partial t)\delta \sim 10^{-28} \text{ m}^{-1} \times L$, far below the required accuracy. We also neglect the δ^2 terms since their contribution together with c^{-4} terms of δ is of an order below 10^{-24} for an altitude difference of the fibre endpoints below 10 km on the Earth's surface. Using these approximations we obtain the following formula for the computed correction

$$\Delta = (\delta_{I0} + \delta_{I2})/2 - \delta_{F1} + \frac{1}{2} \int_0^L \left[\frac{\partial \Omega_+}{\partial t}(t_1, l) + \frac{\partial \Omega_-}{\partial t}(t_1, l) \right] dl \quad (\text{A.35})$$

where the indices $I0, I2$ mean that the quantity is evaluated at the position of the observer I at the time t_0 or t_2 , respectively, and similarly the index $F1$ means the quantity is evaluated at the position of the observer F and time t_1 . In (A.35) we can approximate

$$(\delta_{I0} + \delta_{I2})/2 \approx \delta_{I1} \quad (\text{A.36})$$

since the non-linearity of $\delta_I(t)$ causes deviations which are far below the required accuracy.

Inserting (A.36) and the Ω_{\pm} given by (A.13) up to the c^{-3} order into (A.35) we obtain the formula (20).

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